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權益連結型保單解約權之合理定價

Valuation of the Surrender Option Embedded in Equity-Linked Life Insurance

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摘要

本研究旨在估算具保障收益之權益連結型保單解約權的合理價值。Brennan ＆Schwartz (1976,1979)首先利用選擇權定價理論估算具保障收益之權益連結型保單的合理保費。本文在 Brennan＆Schwartz 的定價模型下，將保單條款中的解約權納入考慮，估算具保障收益且附有解約權之權益連結型保單的合理保費。本研究結合選擇權定價理論與傳統保險精算原理，推導出具保障收益且附有解約權之保單的理論價值，進而得出解約權之理論價值。發現此解約權之價值與解約價值及保障收益存有特定之關係，並說明了解約價值及保障收益如何影響解約權之價值。本研究結果可做為實務上合理保費訂定之參考依據。

關鍵字：權益連結型保單、解約權、保障收益、解約價值。

1. Introduction

Compared to traditional insurance products, one distinguishing feature of equity-linked life insurance contracts is that the benefit payable at expiration depends upon the market value of some reference portfolio. This portfolio may consist of stocks, bonds, and other financial assets with mutual funds as typical cases. Because an insured has to bear more risk for this feature, insurance companies have to enhance their investment products by additional insurance feature. Thus they typically provide policyholder with a minimum benefit or asset value guarantee on death of the insured or maturity of the contract. This kind of insurance product is called the equity-linked life insurance policies with an asset value guarantee.

Since equity-linked insurance contract is characterized by a random amount of benefit, which is linked to some financial asset, the modern financial valuation techniques are needed to value such an insurance product. The pioneer treatments of equity-linked contracts with guarantees by modern financial valuation techniques are conducted by Brennan＆Schwartz (1976,1979) and Boyle＆Schwartz (1977). The case of an Equity-Linked Endowment Policy with an Asset Value Guarantee (ELEPAVG henceforth) is studied in their works. They recognized that the payoff from an equity-linked insurance policy at expiration is identical to the payoff from a European call option plus a certain amount (the guaranteed amount) or to the payoff from a European put option plus the value of the reference portfolio. Assuming that the market value of the reference fund follows the geometric Brownian motion, they utilized the option pricing model of Black＆Scholes (1973) to value the equity-linked endowment policy and obtained a close-form solution for the single premium contract. Additionally, the more recent works are based on the martingale pricing theory as Delbaen (1990), an extension of the Black-Scholes model by Harrison and Kreps (1979).

Different researches on equity-linked life insurance are rather abundant. A variety of different equity-linked insurance products have been discussed. For example, subsequent works considering different structure of benefit, including

the caps in Eker and Persson (1996) and the endogenous minimum guarantees in Bacinello and Ortu (1993), were carried out. Additionally, Nielsen and Sandmann (1995) and Bacinello and Persson (2002) incorporated stochastic interest rates into their pricing models.

Adopting the models of Brennan & Schwartz (1976,1979) for endowment policies, we take a one-shot surrender option embedded in the insurance contracts into account in this work. The distinguishing feature of ELEPAVG with the surrender option is to offer policyholders a right to surrender any portion of fund value at specified time in term of contract. The one-shot surrender option, that a policyholder can exercise the option only once, will be studied. The value of the additional surrender option with single premium case is derived. An optimal surrender behavior for ELEPAVG with the surrender option is explored first. Then based on the optimal behavior of a policyholder, the fair single premium of such the insurance contracts charged by issuers is calculated and the pricing formula of the one-shot surrender option is also derived.

The content of this paper is organized as follows. In section 2, the notations and assumptions applied in our valuation framework are presented. The pricing formulas of single premium for two types of the equity-linked life insurance product, i.e. pure endowment contracts and term insurance contracts, are also reviewed in this section. The investment considerations at the initiation of the contract and the optimal surrender behavior for ELEPAVG with the one-shot surrender option are described in section 3. Additionally, the fair price of the surrender option embedded in the equity-linked insurance policies is also derived. The two-period case is discussed in details in section 4, which would provide some more intuitive implications. Finally, the conclusive remarks are provided in section 5.

2. Notation and definition of the contract

This paper focuses on the valuation of ELEPAVG incorporating the one-shot surrender option. For simplicity, it is concentrated on the case where the

policyholder can exercise the surrender option only at a certain time point in the term of the contract. The financial asset process and the insured's death process would be defined first. The valuation framework for equity-linked insurance policies is then introduced. To make the expression more concise, the martingale approach to contingent claim valuation is applied to price all insurance products discussed here.

2.1 The financial assets

The equity-linked life insurance policy is affected by both financial risk and mortality. Financial environment is set–up first, and then the structure of the contract we consider is stated. As contrasted with traditional insurance product, the benefit of the equity-linked insurance is random. To describe the feature, the value of such an insurance product is modeled with a stochastic process. As proposed by Brennan＆Schwartz (1976), it is assumed that the price process of reference fund follows a geometric Brownian motion with volatility parameter (σ) while the interest rate (r) is assumed to be deterministic. We consider a continuous trading economy with a time interval [0, T]. The uncertainty is characterized by the probability space (Ω, F, Q), where Ω is the state space, F is the σ-algebra and Q represents the equivalent martingale measure. The process, S_t, is adapted to the filtration I_t of the Brownian motion. All trades are assumed to take place in a frictionless market, i.e. there are no transaction costs or taxes.

2.2 Mortality factor

To price an insurance product, the mortality factor should be taken into account. The principle of equivalence, which is traditional actuarial method, does not deal with random benefit. Typically, financial valuation theories are used together with the principle of equivalence to price the equity-linked insurance product under the assumptions that the mortality is stochastically independent from the financial risk and the insurer is risk-neutral with respect to mortality.

The contract considered is a single–premium equity-linked life insurance with guarantees issued at time 0 and maturing T years later. The time horizon T is

divided into n periods. Each subinterval is denoted by Δ. Thus, $n\Delta$ is equal to T. we assume that the death probabilities in each subinterval exist between time interval [0,T]. For the age of an insured x, the mortality distribution can be extracted from a mortality table. Let the mortality of the t–th period be denoted by ${}_tq_x$, t = 1,2,…, n, for an insured of the age x. Since the financial market process and the insured's death process have been defined, the pricing work for equity-linked insurance could be done.

2.3 Single premium of equity-linked insurance contracts

Before considering the surrender option, we first review the case without the surrender option. In this subsection, we introduce the calculation for the single premium of ELEPAVG in general form. Basically, there are two types of equity-linked life insurance policies, which are similar to traditional pure endowment, maturing upon survival at the term of the contract, and term insurances, maturing upon death before the term of the contract. The pricing formula of the single premium discussed here will be utilized to value ELEPAVG with a one-shot surrender option.

It is supposed that S_t is the unit market price of reference mutual fund at time t(or at the end of t-th period). The minimum asset guarantee of per share at time t G_t is denoted by G_t, which is a function of the initial value of reference fund and time. A policyholder is assumed to make a single investment amount of $m_0 \times S_0$ into the fund at time 0, where m_0 is the units invested in reference portfolio at time 0. If the contract matures at time t, the benefit receivable at time t depends on the market value of the fund at time t or on the guarantee value, i.e. Max (S_t,G_t). Based on the integration of the financial valuation theories and the principle of equivalence, we can calculate the single premium of the equity-linked life insurance contracts.

First, the single premium of the equity-linked pure endowment contract with a guarantee maturing at time T represented by $\pi_{(p)}$ is

$$\pi_{(p)} = {}_n p_x E^Q [\, e^{-r\,n\Delta} \max(S_T, G_T) \mid I_0] \tag{1}$$

where ${}_n p_x = (1 - {}_1 q_x)(1 - {}_2 q_x)\cdots(1 - {}_n q_x)$ is the probability that the insured is still alive at time t = n and $E^Q[\,\cdot \mid I_t]$ is the conditional expectation operator with respect to a equivalent martingale measure Q and a filtration I_t..

Similarly, the single premium of the equity-linked term insurance contract with a guarantee maturing at time T represented by $\pi_{(t)}$ is

$$\pi_{(t)} = \sum_{t=1}^{n} {}_{t-1|}q_x E^Q [\, e^{-r\,t\Delta} \max(S_t, G_t) \mid I_0] \tag{2}$$

where ${}_{t-1|}q_x = (1 - {}_1 q_x)(1 - {}_2 q_x)\cdots(1 - {}_{t-1} q_x)\, {}_t q_x$ is the probability that the insured dies within the t-th period.

Using (1) and (2), the market price at time 0 of the equity-linked endowment insurance policies represented by $\pi_{(e)}$, which put a equity-linked pure endowment contract and a equity-linked term contract together, can be written by

$$\pi_{(e)} = {}_n p_x E^Q [e^{-r\,n\Delta} \max(S_T, G_T) \mid I_0] + \sum_{t=1}^{n} {}_{t-1|}q_x E^Q [e^{-r\,t\Delta} \max(S_t, G_t) \mid I_0] \tag{3}$$

The pricing formula of (3) will be utilized to evaluate the ELEPAVG with a one-shot surrender option in the following sections.

3. Valuation for ELEPAVG with a surrender option

ELEPAVG with a one-shot surrender option is considered in this section. The work is restricted to the case which a policyholder can exercise the surrender option only once. Suppose a policyholder is permitted to exercise the surrender option only at time t = k (0<k<n). The cash surrender value (exercise price) is set by $W(S_k)$, which is always a function of S_t. It is assumed that surrender proportion of fund value is not limited. Furthermore, the purpose for a

policyholder's surrender decision is to maximize the market value of the life insurance contract.

3.1 Investment consideration at initiation of contract

We suppose that the policyholder initially plan to surrender m* shares of the fund at time k. In another word, if the insured is alive at time k, he will surrender m^* shares of the fund whatever the realized value of S_k will be. According to his surrender plan, the market value at time 0 of such an endowment insurance contract may be written by

$$\pi_{(e,u)}$$

$$= m_0 \{ \sum_{t=1}^{k} {}_{t|}q_x E^Q [e^{-r t\Delta} \max(S_t, G_t) \mid I_0] + \xi (1 - \sum_{t=1}^{k} {}_{t|}q_x) V_0(W(S_k))$$

$$+ (1-\xi)(\sum_{t=k+1}^{n} {}_{t|}q_x E^Q[e^{-r t\Delta t} \max(S_t,G_t) \mid I_0] + {}_n p_x E^Q [e^{-r n\Delta t} \max(S_T,G_T) \mid I_0])\} \quad (4)$$

where $\xi = m^*/ m_0$, called surrender ratio.

According to (4), the policyholder will try to find the optimal amount of q to maximize the market value of the contract. However the insured can't determine the optimal value of q at the initial time. It is because he does not know what value of $W(S_k)$ will become. In another word, the optimal value of q is contingent to different situations. The criterion by which the policyholder chooses the optimal surrender ratio is discussed in the next subsection.

3.2 The optimal surrender behavior

Following previous subsection, we need to find the optimal surrender behavior for a rational policyholder. The optimal surrender behavior determines the surrender ratio. It is also assumed that policyholder's decision making is based on maximization of the market value of the insurance contract at time k. Thus

the problem to be solved is to find the optimal surrender ratio q*, which would maximize the market value of the contract at time k, subject to $0 \leqq \xi^* \leqq 1$.

Similar to (3), if the policyholder is alive at time k, the market values at time k of the insurance contract, denoted by $\pi_{(e,k)}$, may be written by

$$\pi_{(e,k)}$$

$$= m_0\{\xi\, W(S_k) + (1-\xi)\,(\sum_{t=k+1}^{n} {}_{t|}q_x\, E^Q[e^{-r\,t\Delta}\max(S_t,G_t) \mid I_k]$$

$$+ {}_np_x\, E^Q\,[e^{-r\,n\Delta}\max(S_T,G_T) \mid I_k]\,)\,\}$$

$$= m_0\{\xi\, W(S_k) + (1-\xi)\,\Theta(S_k)\} \quad (5)$$

where $\Theta(S_k)$ denote $\sum_{t=k+1}^{n} {}_{t|}q_x\, E^Q[e^{-r\,t\Delta}\max(S_t,G_t) \mid I_k] + {}_np_x\, E^Q\,[e^{-r\,n\Delta}\max(S_T,G_T) \mid I_k]$, which is a function of S_k.

To maximize (5), the optimal surrender ratio ξ^* which a policyholder will take would be determined by comparing $W(S_k)$ with $\Theta(S_k)$. The optimal surrender ration ξ^* depends on the realized value of S_k. As $W(S_k) > \Theta(S_k)$, the optimal surrender ration ξ^* will equal one. As $W(S_k) < \Theta(S_k)$, the optimal surrender ration ξ^* will be equal to zero.

A break-even point, $\bar{S}_k$, could be found such that $W(\bar{S}_k)$ is equal to $\Theta(\bar{S}_k)$. The criterion of the surrender decision for a policyholder, which depends on S_k, is summarized by following expression.

$$\xi^* = \begin{cases} 1 & \text{as } S_k > \bar{S}_k \\ 0 & \text{as } S_k < \bar{S}_k \end{cases} \quad (6)$$

3.3 Pricing the surrender option

The fair single premium of ELEPAVG with a one-shot surrender option is calculated in this subsection. Since the insured's behavior follows the optimal investment strategy of (6), the break-even point, $\overline{S}_k$, will divide the distribution of S_k into "exercise" and "no exercise" region. The payoff at time k depends on which situation occurs. There are four situations, which are determined by two factors, i.e. the realized value of S_k and the survival status of insured in k-th period. The payoff per share in each situation is summarized in Table 1.

Table 1: The payoffs in the four situations at time t = k

		The status of insured in k-th time interval	
		Dead	Alive
The value of S_k	$S_k > \overline{S}_k$	Situation 1: Max (S_k, G_k)	Situation 2: W(S_k)
	$S_k < \overline{S}_k$	Situation 3: Max (S_k, G_k)	Situation 4: 0

According the payoff in each situation, the market value at time 0 of ELEPAVG with the surrender option represented by $\pi_{(e,s)}$ may be derived by

$$\pi_{(e,s)}$$

$$= m_0 \{ \sum_{t=1}^{k-1} {}_{t|}q_x E^Q [e^{-r t\Delta} \max(S_t, G_t) \mid I_0] + {}_{t|}q_x E^Q [e^{-r k\Delta} \max(S_k, G_k) \mid I_0]$$

$$+ (1- \sum_{t=1}^{k} {}_{t|}q_x)[E^Q (e^{-r k\Delta} W^*(S_k) \mid I_0) + E^Q (e^{-r k\Delta} \Theta^*(S_k) \mid I_0)] \} \quad (7)$$

Where $W^*(S_k) = W(S_k)$, if $S_k > \overline{S}_k$ and $W^*(S_k) = 0$, if $S_k < \overline{S}_k$;

$\Theta^*(S_k) = \Theta(S_k)$, if $S_k < \overline{S}_k$ and $\Theta^*(S_k) = 0$, if $S_k > \overline{S}_k$.

By making (7) minus (3), the fair market value of one-shot surrender option embedded in equity-linked insurance policies represented by $\pi_{(s)}$ may be derived by

$$\pi_{(s)} = m_0 \{ {}_k p_x [E^Q (e^{-r k\Delta} W^*(S_k) \mid I_0) + E^Q (e^{-r k\Delta} \Theta^*(S_k) \mid I_0)]$$

$$- \sum_{t=k+1}^{n} {}_{t|}q_x E^Q [e^{-r t\Delta} \max(S_t, G_t) \mid I_0] - {}_n p_x E^Q [e^{-r n\Delta} \max(S_T, G_T) \mid I_0]\} \quad (8)$$

To make the expression more concise, (8) may be rewritten by

$$\pi_{(s)} = m_0 (\Gamma - \Sigma) \quad (9)$$

where Γ and Σ denote ${}_k p_x [E^Q (e^{-r k\Delta} W^*(S_k) \mid I_0) + E^Q (e^{-r k\Delta} \Theta^*(S_k) \mid I_0)]$ and $\sum_{t=k+1}^{n} {}_{t-1|}q_x E^Q [e^{-r t\Delta} \max(S_t, G_t) \mid I_0] + {}_n p_x E^Q [e^{-r n\Delta} \max(S_T, G_T) \mid I_0]$ respectively.

According to (9), it is not necessary that the value of $\pi_{(s)}$ is positive since it is not absolute that Γ is larger than Σ. Thus, whether the value of one-shot surrender option is positive depends on how to set the cash surrender value (exercise price function), $W(S_k)$, and the guarantee function, G_t. In another word, both the cash surrender option and the guarantees provided by insurer will affect insured's intention to pay the premium for such the additional surrender option. If the value of (9) is not positive, the surrender option is not valuable and the premium of ELEPAVG with the surrender option may be as same as that of ELEPAVG with no surrender option.

4. Implications on two-period case

The two-period case is discussed in details in this section. Let the term of insurance contract divided into two subintervals. Three time points, i.e. time 0, time1 and time2, are considered now. The surrender option may be exercised only at time 1.To make the model simpler, it is assumed that the asset value guarantee is constant, i.e. $G_t = g$ and the cash surrender option, $W(S_1)$, is set by a linear function of S_1, i.e. $(1+\varepsilon)S_1$, where ε is a constant.

Now we also suppose that the policyholder initially have a preferred surrender

plan. The surrender portion of reference fund shares is ξ. Even the results in previous section imply that the policyholder can't determine the optimal surrender portion at initiation of the contract. Given the surrender plan, the market value of the insurance contract denoted by $\pi_{(2)}$ may be written by

$$\begin{aligned}\pi_{(2)} &\\ = m_0\{ {}_1d_x[e^{-r\Delta} g + C(S_0,\Delta,g)] &+ \xi (1-{}_1d_x)V_0((1+\varepsilon)S_1) \\ &+ (1-\xi)(1-{}_1d_x)[e^{-r2\Delta} g + C(S_0,2\Delta,g)]\} \qquad (10)\end{aligned}$$

where $C(S_0,\Delta,g)$ denotes the European call option value with the current price of underlying asset, S_0, the time to maturity, Δ, and the exercise price, g; and $V_t(S_\tau)$ denotes the market value of S_τ at time t

The premium charged by an issuer to provide both guarantee and surrender options, denoted by α, is given by

$$\begin{aligned}\alpha &= \pi_{(2)} - m_0S_0 \\ &= m_0\{{}_1d_x[e^{-r\Delta} g + C(S_0,\Delta,g) - S_0] + \xi (1-{}_1d_x)V_0((1+\varepsilon)S_1) - (1-{}_1d_x)S_0 \\ &\quad + (1-\xi)(1-{}_1d_x)[e^{-r2\Delta} g + C(S_0,2\Delta,g)]\} \qquad (11)\end{aligned}$$

Based on the Martingale pricing theory, i.e. $E^Q[e^{-\Delta r}S_1] = S_0$, $V_0(S_1)$ is equal to S_0. Thus (10) may be rewritten by

$$\pi_{(2)} = m_0\{S_0 + {}_1d_x\, P(S_0,\Delta,g) + (1-\xi)(1-{}_1d_x)\, P(S_0,2\Delta,g) + \xi\varepsilon S_0\} \qquad (12)$$

where $P(S_0,\Delta,g)$ denotes the European put option value with current price of underlying asset, S_0, time to maturity, Δ, and the exercise price, g.

Note that (12) with q = 0 is a special case of Brennan and Schwartz's general pricing formula. According (12), it is obvious that the market value of the contract is an increasing function of g and ε.

As the results derived in the third section, the optimal surrender behavior as of

time 1 depends on the realized value of S_1. The break-even point, $\bar{S}_1$, must satisfy the following equations.

$$\bar{S}_1 + P(\bar{S}_1, \Delta, g) = (1+\varepsilon)\ \bar{S}_1 \qquad (13)$$

or

$$\varepsilon = P(\bar{S}_1, \Delta, g) / \bar{S}_1 \qquad (14)$$

(13) or (14) implies that the value of ε can not allowed to be too small. If ε is smaller than $P(S_1, \Delta t, g) / \bar{S}_1$ for each S_1, such a surrender option become valueless. Therefore, the larger the guarantee g is, the more the fair premium that the insurer will charge should be, which. The more the guarantee offered by insurer is, the more the cash surrender value should be. Otherwise, the surrender option may not provide the additional value for a policyholder. For the extreme case, the exercise price function with ε = 0 will make the surrender option valueless. Thus, given a certain single premium for the equity-linked insurance policies, the relationship between guarantee value and cash surrender value should be specified. The implications of the result may be helpful for both insurance product designer and insurance buyer.

5. Conclusion

The key feature of equity-linked life insurance policies is the uncertainty of the future insurance benefit. Our work has shown how to use the Martingale pricing theory and principle of equivalence to price the one-shot surrender option embedded in equity-linked life insurance contracts. The optimal surrender behavior for the policyholder is found. Then the single premium of ELEPAVG with the surrender option, which is charged fairly by issuers, has been derived. Further the pricing formula for the valuation of the surrender option is also found. The result of this research provides some implications in practice for both insurer and investor. This study could be extended to the cases with multiple surrender options or American-type surrender option.

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