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An Extreme Value Theory Approach for Analyzing the Extreme Risk of the Gold Prices

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Abstract

Financial markets frequently experience extreme movements in the negative side. Accurate computation of value at risk and expected shortfall are the main tasks of the risk managers or portfolio managers. In this paper, gold prices (in US dollars) have been examined to illustrate the main idea of extreme value theory and discuss the tail behaviour. GEV and GPD are used to compute VaR and ES. The results show that GPD model with threshold is a better choice.

Keywords: value at risk, expected shortfall, generalized extreme value distribution, generalized Pareto distribution

1. Introduction

Risk managers and Portfolio managers concern extreme negative side movements in the financial markets. A long list of research has posted on this topic. Ramazan and Faruk (2006) examine the dynamics of extreme values of overnight borrowing rates in an inter-bank money market. Generalized Pareto distribution has been picked for its well fitting. Fernandez (2005) using extreme value theory to the United States, Europe, Asia, and Latin America financial markets for computing value at risk (VaR). One of his findings is, on average, EVT provides the most accurate estimate of VaR. Byström (2005) applied extreme value theory to the case of extremely large electricity price changes and declared a good fit with generalized Pareto distribution (GPD). Bali (2003) determines the type of asymptotic distribution for the extreme changes in U.S. Treasury yields. In his paper, the thin-tailed Gumbel and exponential distribution are worse than the fat-tailed Fréchet and Pareto distributions. Neftci (2000) found that the extreme distribution theory fit well for the extreme events in financial markets. Gençay and Selçuk (2004) investigate the extreme value theory to generate VaR estimates and study the tail forecasts of daily returns for stress testing. Marohn (2005) studies the tail index in the case of generalized order statistics, and declares the asymptotic properties of the Fréchet distribution. Brooks, Clare, Dalle Molle and Persand, G., (2005) apply a number of different extreme value models for computing the value at risk of three LIFFE futures contracts. Based upon the long list of applications of extreme value theory and theory basis, this paper will first discuss the generalized extreme value methods and then fit the model with gold prices (in US dollars) to compute the chance of new record generated in the next period. Different models are also been fitted with gold data. The results show that GPD model with threshold is a better choice.

2. Methodology

2.1 Generalized Extreme Value (GEV) Distribution

Extreme movements in gold prices can be measured by the daily returns of the gold prices per troy ounce (reported on London Gold Market). Losses or negative returns are the main concerns in the field of financial risk management, for examples stock market crashes. For simplicity, extreme movements in the left tail of the distribution can be characterized by the positive numbers of the right high quintiles. Let X_i be the negative of the ith daily return of the gold prices between day i and day $i-1$. Define

$$X_i = -(\ln P_i - \ln P_{i-1})$$

where P_i and P_{i-1} are the gold prices of day i and day $i-1$. Suppose that $X_1, X_2, ..., X_n$ be iid random variables with an unknown cumulative distribution function (CDF) $F(x) = \Pr(X_i \le x)$. Extreme values are defined as maxima (or minimum) of the n independently and identically distributed random variables $X_1, X_2, ..., X_n$. Let $X_{(n)}$ be the maximum negative side movements in the daily gold prices returns, that is

$$X_{(n)} = \max(X_1, X_2, ..., X_n).$$

Since the extreme movements are the focus of this study, the exact distribution of $X_{(n)}$ can be written as

$$\Pr(X_{(n)} \le a) = \Pr(X_1 \le a, X_2 \le a, ..., X_n \le a)$$
$$= \prod_{i=1}^{n} F(a)$$
$$= F^n(a) \quad (1)$$

In practice the parent distribution F is usually unknown or not precisely known. The empirical estimation of the distribution $F^n(a)$ is poor in this case. Fisher and Tippet (1928) derived the asymptotic distribution of $F^n(a)$. Suppose $\mu_{(n)}$ and $\sigma_{(n)}$ are sequences of real number location and scale measures of the maximum statistic $X_{(n)}$. Then the standardized maximum statistic

$$Z^*_{(n)} = (X_{(n)} - \mu_{(n)}) / \sigma_{(n)}$$

Converges to $z = (x-\mu)/\sigma$ which has one of three forms of non-degenerate distribution families such as

$$\begin{aligned} H(z) &= \exp\{-\exp[-z]\}, \quad -\infty < z < \infty \\ H(z) &= \exp\{-z^{-1/\xi}\}, \quad z > 0 \\ &= 0, \quad else \\ H(z) &= \exp\{-[-z]^{1/\xi}\}, \quad z < 0 \\ &= 1, \quad else \end{aligned} \quad (2)$$

These forms go under the names of Gumbel, Fréchet, and Weibull respectively. Here μ and σ are the mean return and volatility of the extreme values x and ξ is the shape parameter or called $1/\xi$ the tail index of the extreme statistic distribution. With $\xi = 0, \xi > 0, \xi < 0$ represent Gumbel, Fréchet, and Weibull types of tail behavior respectively. In fact Gumbel, Fréchet, and Weibull types can be fit for exponential, long, and short tails respectively.

Embrechts and Mikosch (1997) proposed a generalized extreme value (GEV) distribution which included those three types and can be used for the case stationary GARCH processes. GEV distribution has the following form

$$\begin{aligned} H_\xi(x;\mu,\sigma) &= \exp\{-\exp[-(x-\mu)/\sigma]\}, \quad -\infty \le x \le \infty; \xi = 0 \\ &= \exp\{-[1+\xi(x-\mu)/\sigma]^{-1/\xi}, \ 1+\xi(x-\mu)/\sigma > 0; \xi \ne 0 \end{aligned} \quad (3)$$

2.2 Parameters Estimation for GEV

Suppose that block maxima $B_1, B_2, ..., B_k$ are independent variables from a GEV distribution, the log-likelihood function for the GEV, under the case of $\xi \ne 0$, can be given as

$$\ln L = -k \ln \sigma - (1 + 1/\xi)\sum_{i=1}^{k} \ln\{1 + \xi(B_i - \mu)/\sigma\} - \sum_{i=1}^{k}\{1 + \xi(B_i - \mu)/\sigma\}^{-1/\xi} \quad (4)$$

For the Gumbel type GEV form, the log-likelihood function can be written as

$$\ln L = -k \ln \sigma - \sum_{i=1}^{k}(B_i - \mu)/\sigma - \sum_{i=1}^{k}\exp\{-(B_i - \mu)/\sigma\} \quad (5)$$

As Smith (1985) declared that, for $\xi > -0.5$, the maximum likelihood estimators, for ξ, μ, and σ, satisfy the regular conditions and therefore having asymptotic and consistent properties. The number of blocks, k and the block size form a crucial tradeoff between variance and bias of parameters estimation.

POT and GPD

Fitting models with more data is better than less, so peaks over thresholds (POT) method utilizes data over a specified threshold. Define the excess distribution as

$$F_h(x) = \Pr(X - h < x \mid X > h) = \frac{F(x+h) - F(h)}{1 - F(h)} \quad (6)$$

where h is the threshold and F is an unknown distribution such that the CDF of the maxima will converge to a GEV type distribution. For large value of threshold h, there exists a function $\tau(h) > 0$ such that the excess distribution of equation (6) will approximated by the generalized Pareto distribution (GPD) with the following form

$$H_{\xi,\tau(h)}(x) = 1 - \exp(-x/\tau(h)), \quad \xi = 0$$
$$= 1 - (1 + \xi x/\tau(h))^{-1/\xi}, \xi \neq 0 \quad (7)$$

where $x > 0$ for the case of $\xi \geq 0$, and $0 \leq x \leq -\tau(h)/\xi$ for the case of $\xi < 0$. Define $x_1, x_2, ..., x_k$ as the extreme values which are positive values after subtracting threshold h. For large value of h, $x_1, x_2, ..., x_k$ *is* a random sample from a GPD, so the unknown parameters ξ and $\tau(h)$ can be estimated with maximum likelihood estimation on GPD log-likelihood function.

VaR and ES

Based on equation (6) and GPD distribution, the unknown distribution F can be derived as

$$F(y) = (1 - F(h))H_{\xi,\tau(h)}(x) + F(h) \quad (8)$$

where $y = h + x$. $F(h)$ can be estimated with non-parametric empirical estimator

$$\hat{F}(h) = (n-k)/n$$

where k is the number of extreme values exceed the threshold h. Therefore the estimator of (8) is

$$\hat{F}(y) = (1-\hat{F}(h))\hat{H}(x;\hat{\xi},\hat{\tau}(h)) + \hat{F}(h) \tag{9}$$

where $\hat{\xi}$ and $\hat{\tau}(h)$ are mle of GPD log-likelihood. High quantile VaR and expected shortfall can be computed using (9). First, define $F(VaR_q) = q$ as the probability of distribution function up to q^{th} quantile VaR_q. Therefore

$$\begin{aligned}\hat{VaR}_q &= \hat{F}^{-1}(q) \\ &= h + \hat{\tau}(h)\{[(n/k)(1-q)]^{-\hat{\xi}} - 1\}/\hat{\xi}\end{aligned} \tag{10}$$

Next, given that VaR_q is exceeded, define the expected loss size, expected shortfall (ES), as

$$\begin{aligned}ES_q &= E(X \mid X > VaR_q) \\ &= VaR_q + E(X - VaR_q \mid X > VaR_q)\end{aligned} \tag{11}$$

From (10), $\hat{ES}_q$ can be computed using $\hat{VaR}_q$ and the estimated mean excess function of GPD distribution. Therefore,

$$\hat{ES}_q = \hat{VaR}_q/(1-\hat{\xi}) + (\hat{\tau}(h) - \hat{\xi}\, h)/(1-\hat{\xi})$$

3. Data and Empirical Results

3.1 Data and Data Exploration

The upper panel of Figure 1 shows the daily gold prices per troy ounce (in US dollars) over the period of January 1, 1985 through March 31, 2006. The lower panel of Figure 1 is the continuous percentage returns of the gold prices in the upper panel. Table 1 shows that daily gold returns have a positive skew (0.76>0) and large kurtosis (14.16>3). It shows the distribution of returns has a fat tail.

Figure 1 Daily Gold Prices Per Troy Ounce (in US Dollars) and Daily Gold Percentage Returns

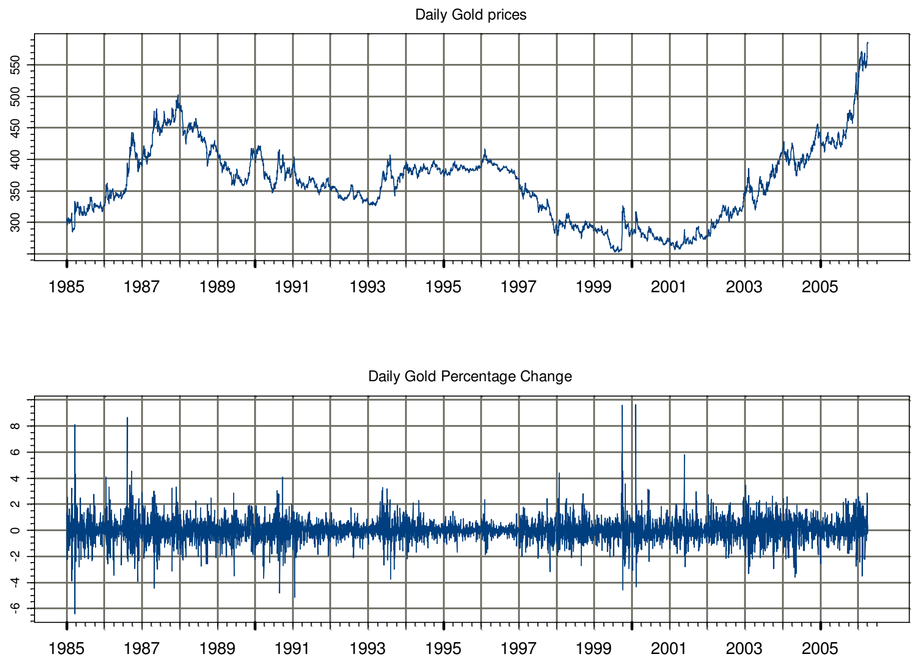
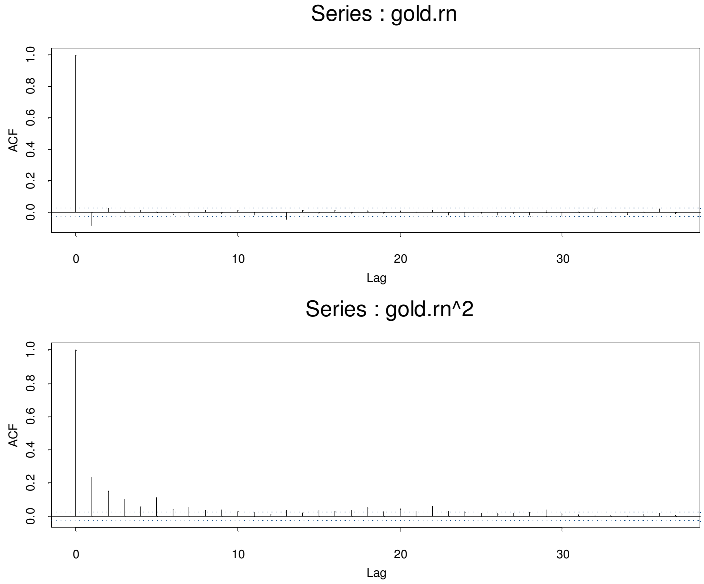


Table 1 Descriptive Statistics of Daily Gold Percentage Returns

Minimum	-6.43
1st Quarter	-0.43
Mean	0.012
Median	0
3rd Quarter	0.43
Maximum	9.64
Total N	5371
Std Deviation	0.90
Skewness	0.76
Kurtosis	14.16

Figure 2 shows that there is no autocorrelation in the daily gold percentage return series but the squared daily gold percentage returns appear significant autocorrelation. Figure 2 suggest that the daily gold percentage returns can be a GARCH time series model. It is reasonable to apply GEV distribution to model the maxima negative returns and GPD distribution to model the excess distribution for negative daily gold percentage returns.

Figure2 ACF of Daily Gold Percentage Returns and Squared Daily Gold Percentage Returns



3.2 Parametric Maximum Likelihood Estimates

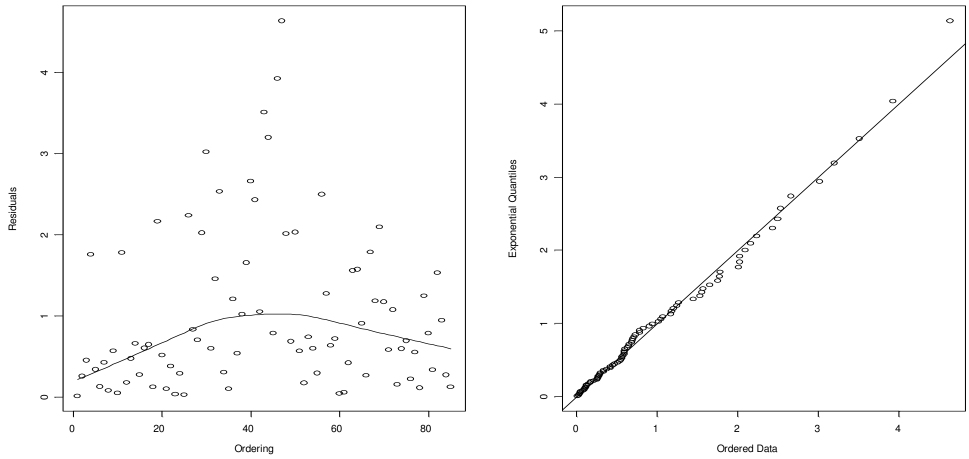
The block maxima of negative returns have been fitted to GEV model with three different block sizes (yearly, quarterly, and monthly). The results, listed in Table 2, show the estimates of three parameters ξ, σ, and μ with increasing block numbers and the estimates of standard error of the parameters are listed in parentheses. All block numbers show that shape parameter is positive except yearly block size case is negative but not significant. All the parameters have decreasing estimated standard errors as the number of blocks increased. In general, the block maxima of the negative returns follows a Fréchet family of GEV for quarter and month block frames. The Fréchet type of GEV confirms that the original series has fat tail.

Table 2 Parametric Maximum Likelihood Estimates with three block frames (block sizes: year, quarter, and month) and chance of a new record will occur during the next period

	Year	Quarter	Month
ξ	-0.06 (0.19)	0.11 (0.10)	0.17 (0.06)
σ	0.97 (0.18)	0.75 (0.07)	0.59 (0.03)
μ	2.92 (0.24)	1.72 (0.09)	1.17 (0.04)
New record	1.77%	0.86%	0.44%

Figure 3, from left to right is the crude residual plot and quantile-versus-quantile plot with exponential distribution as the reference distribution for quarter block maxima. The Q-Q plot looks linear and shows that the GEV distribution model is a good choice. It shows that the Fréchet distributions are fitted well for the block maxima negative returns.

Figure 3　Residual plots versus quarter frame block maxima



With the above parameters and corresponding Fréchet distributions, there is 1.77%, 0.86% and 0.44% chance that a new record will be occurred during the next period (year, quarter and month). The point estimates and 95% confidence interval of 10 and 20 years return levels are listed below. Given the same time interval (10 or 20 years), the point estimate is increased with decreasing block sizes. With the increasing time interval, the point estimate increased and confidence interval is wider. It reveals that time is a risk factor.

Table 3　Return levels for the three block frames with 95% confidence bounds

	Year	Quarter	Month
10 year return level	4.98 (4.31, 6.91)	5.13 (4.28, 7.17)	5.54 (4.57, 7.29)
20 year return level	5.58 (4.75, 9.13)	5.96 (4.77, 9.08)	6.53 (5.18, 9.11)

3.3 Extreme Over Thresholds and Risk Measures

A suitable threshold should be specified to find the GPD approximation of the negative returns' excess distribution. Try several thresholds to fit the excess GPD. Do QQ-plots with corresponding GPD distributions for the daily negative returns over the thresholds (shown on Figure 4) and compare some significant numbers (Table 4).

Figure 4 Residual diagnostic check for GPD (with threshold 2) fit to Daily Gold negative returns

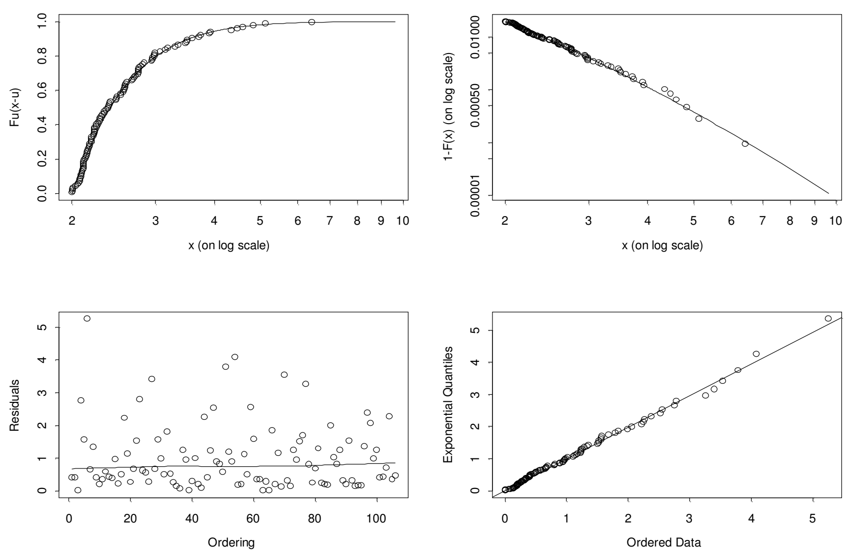


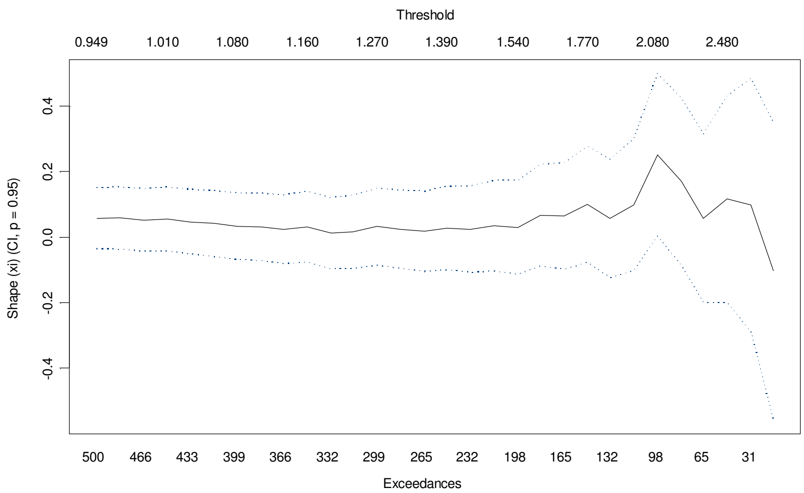
Figure 4 show that the threshold 2 model fit the GPD well. On the top-left panel of Figure 4 is the GPD estimate of the excess distribution, top-right panel is the tail estimate of equation (9), bottom-left panel is the scatter plot of residuals, and bottom-right panel is the QQ-plot of residuals. Base on the diagnostic plots show data fit GPD model well.

Table 4 shows that the shape parameter is stable increasing as threshold increasing from 1.5 to 2, and the shape parameter swings to negative value. The details are displayed as Figure 5. Theoretically, shape parameter of GEV and GPD should be the same. The shape parameters of threshold 2 and the month block size GEV are similar (see table 2). According to the referred information, the estimate value 2 for shape parameter is the best specified threshold. The GPD approximation has shape parameter 0.15 and scale parameter 0.56.

Table 4　GPD model fitting results with thresholds from 1 to 6

threshold	# of exceed	Probability less than threshold	ξ	$\tau(h)$
1	461	0.9142	0.06 (0.05)	0.62 (0.04)
1.25	306	0.9430	0.03 (0.06)	0.66 (0.05)
1.5	210	0.9609	0.02 (0.07)	0.68 (0.06)
1.75	152	0.9717	0.09 (0.09)	0.60 (0.07)
2	106	0.9803	0.15 (0.12)	0.56 (0.09)
2.25	63	0.9882	0.05 (0.14)	0.72 (0.14)
2.5	46	0.9914	0.10 (0.18)	0.67 (0.16)

Figure 5　Estimates of shape parameter for negative returns as a function of the threshold values.



To measure the risk, Value-at-Risk and expected shortfall (ES) can be calculated based on GPD approximation with threshold 2. The results is listed in row 1 of Table 5. For comparison, the values of the GPD approximation with threshold 2 and normal distribution approximations are listed in the following table. All the values of Normal Distribution approximation is under estimate. GPD model with threshold 2 is a better chosen.

Table 5 Risk measures (VaR and ES) for GPD model with thresholds 2 compared with normal distribution

	$VaR_{.95}$	$ES_{.95}$	$VaR_{.99}$	$ES_{.99}$
Threshold=2	1.511492	2.085664	2.401366	3.126575
Normal Distr.	1.461313	1.835598	2.071742	2.375272

4. Concluding Remarks

Fisher and Tippet theorem has an assumption of iid, however, the GEV is suitable for stationary time series including stationary GARCH cases. Exploring the dependence properties on extreme values and the joint-tail properties of multivariate extreme value cases are appeared not long ago. Further investigation is definite welcome.

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以極值理論方法分析金價之極端風險

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摘要

金融市場經常面臨負向的極端移動。風險經理或資產組合經理的主要工作在於精確計算風險值及預期損失。本文以金價(以美元計價)爲例說明極值理論的主要意義及尾端特性並以一般化極值分配及一般化柏拉圖分配計算風險值及預期損失，結果顯示具門檻之一般化柏拉圖分配是較佳選擇。

關鍵字:風險值，預期損失，一般化極值分配，一般化柏拉圖分配