

A Note on the Feynman-Kac Formula and the Pricing of Defaultable Bonds

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Abstract

The Feynman-Kac Formula offers an intuitive approach to solve PDE of financial assets. Traditionally, it is used to model financial assets without default risk. This paper demonstrates the usefulness of Feynman-Kac formula for pricing certain corporate bond models by revisiting Cathcart and El-Jahel (1998) and Schobel (1999). In the first model, a closed-form formula is derived to replace Cathcart and El-Jahel's (1998) original numerical inversion of Laplace transformation for pricing defaultable bonds. In the second model, a simple expectation operation is used to replace Schobel's (1999) original procedure of employing the heat equation and the Green function.

Keywords: default risk, Feynman-Kac formula, forward martingale measure, reduced-form model, structural model

1 Introduction

The traditional pricing methods of securities assume no default risk. Yet the increasing use of over-the-counter securities and the increasing cases of bankruptcy in large firms deem it necessary to incorporate default risk into security pricing models.

Default risk or credit risk modeling is generally classified into two basic approaches: structural models and reduced-form models. The structural approach dates back to Merton (1974). It assumes that the dynamics for the asset value of a firm follows a diffusion process, and that defaultable securities can be treated as a contingent claim on the assets of the firm. For this reason, the structural approach is also referred to as the firm value approach or the option-theoretic approach. Subsequent research following Merton is considerable. Black and Cox (1976) generalize Merton's model in several respects. They allow early default before maturity, incorporate safety covenants, and differentiate senior and junior debts. Longstaff and Schwartz' (1995) paper was one of the first works to assume a stochastic interest rate process, and Zhou (2001) incorporates a jump process for the original diffusion process of the underlying asset value. Despite the immense endeavor to extend Merton's model, these models have failed to explain credit spreads observed in other empirical works; for example, Jones, Mason and Rosenfeld (1984), and Kim, Ramaswamy and Sundaresan (1993).

The reduced-form model, which is also called the hazard-rate model or the intensity-based model, was pioneered by Pye (1974) and then advanced by Jarrow and Turnbull (1995), as well as by Duffie and Singleton (1999). The reduced-form approach does not link the default event to firm value in an explicit way, but rather models default as a stopping time of some given hazard rate process. This approach successfully generates non-zero short-term credit spreads, which better conform to empirical observations. Its main drawback is that the default event lacks economic interpretation concerning its fundamentals.

From a more practical point of view, corporate bonds are characterized by various attributes such as: safety covenants, credit ratings and recovery scheme. Basically, default implies that the bondholder take over the firm. Exogenous bankruptcy (default) refers to the case when default is specified in form of some protective covenants. For example,

default is triggered once the firm value hit an exogenously specified value, for instance, the principal amount of the debt. Most researches, including this paper, discuss issues related to exogenous default. The notion of endogenous bankruptcy (default) covers the situations when bankruptcy is declared by the stockholders. Leland and Toft (1996) among others explore the issues of optimal capital structure within the framework of endogenous bankruptcy.

A firm's credit rating is a measure of the firm's propensity to default. This information is typically released by commercial rating agency, such as Standard & Poor's. An enhanced variant of reduced-form models is to incorporate migrations between credit rating classes. Notable works in this branch of study include Jarrow, Lando and Turnbull (1997) and others.

The recovery rate is the proportion of the "claimed amount" received by the debtholder in case of default. Although the recovery rate is frequently assumed constant and is denoted as δ , there are three different scenarios to define the "claimed amount": the recovery of Treasury, the recovery of face value and the recovery of market value. The recovery of Treasury scheme defines the "claimed amount" as the value of an otherwise equivalent default-free bond, the recovery of face value scheme defines it as the face value, whereas the recovery of market value scheme usually defines it as the pre-default value of the corporate bond. These rules are formally explicated later. One of the main contribution of this research is to show that some corporate bond pricing methods can be much simplified when they are adopting the recovery of Treasury scheme. The key to simplified these models is by exploiting the Feynman-Kac Formula, which is traditionally used to model financial assets without default risk.

The main object of this research is to demonstrate that this theorem can also be used to derive the pricing formula of risky corporate bonds when the recovery scheme at default is assumed to be the recovery of Treasury scheme. In contrast, some researchers, e.g. Cathcart and El-Jahel (1998), evaluate the corporate bond by the numerical method of inverse Laplace transformation, whereas Schobel (1999) derive the closed-form pricing formula of corporate bonds by making use of the heat equation and the Green function. We demonstrate the usefulness of Feynman-Kac formula for pricing certain corporate bond model by revisiting Cathcart and El-Jahel (1998) and Schobel (1999).

The article is organized as follows. In the next section, notation and model setting are formally provided. Section 3 will present a closed-form formula by using the Feynman-Kac Formula to replace Cathcart and El-Jahel's (1998) original numerical inversion of Laplace transformations for pricing the defaultable bond. Section 4 will illustrate how the Feynman-Kac Formula save the burden of using the heat equation and the Green function to derive the pricing formula of Schobel (1999). Finally, a brief conclusion is presented in Section 5.

2 The Framework

Let $X_t \in R^n$ be the vector of state variables at the time of t , with $0 \leq t \leq T < \infty$. The decision horizon T is assumed to be fixed throughout this paper, hence we shall not treat it as a dependent variable in all of the functions related to T . Following the standard assumptions concerning financial stochastic processes, the underlying probability space (Ω, F, P) is supposed to be complete and the augmented filtration $\{F_t : t \geq 0\}$ to be generated by a standard Brownian motion in R^n .

The economy is assumed to be complete, perfect and arbitrage-free. Therefore, we can denote W_t as a standard Brownian motion in R^n under an equivalent martingale measure Q . More specifically, the measure Q is called the spot (risk-neutral) martingale measure, different from the forward martingale measure Q^T introduced later. This fact is due to the fundamental theorem of asset pricing introduced by Harrison and Kreps (1979). We give a brief review of this property in the Appendix.

The state vector X_t is assumed to follow a multi-dimensional stochastic diffusion equation (SDE),

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t \quad (1)$$

where $\mu : R^n \times [0, \infty) \rightarrow R^n$ and $\sigma : R^n \times [0, \infty) \rightarrow R^{n \times n}$ are assumed to be regular enough for (1) to have a unique (strong) solution.

Let $F(X_t, t)$ be the price at the time of t of any derived security in the economy maturing at T , with the maturity price:

$$F(X_T, T) = g(X_T). \quad (2)$$

After exploiting the standard argument from stochastic calculus, we have the following Feynman-Kac partial differential equation (PDE)

$$F_t(x, t) + F_x(x, t)^\top \mu(x) + \frac{1}{2} \text{tr}[F_{xx}(x, t) \sigma_i(x) \sigma_j(x)^\top] = R(x)F(x, t), \quad (3)$$

where $R(x)$ denotes the risk-free short-term interest rate (the short rate). The partial derivative F_x is taken as a column vector of functions and the superscript \top is a transposition operator for the corresponding vector or matrix. In addition, the notation $\text{tr}[\cdot]$ is a trace operator for the corresponding matrix. Now we present the famous Feynman-Kac Formula, which appears in all textbooks of stochastic calculus. One of the more accessible textbooks is Klebaner (1998).

Theorem 1 (Feynman-Kac Formula) *Let X_t be a diffusion satisfying SDE (1). If there is a solution to PDE (3) with the boundary condition (2), then the solution is unique and the solution is:*

$$C(x, t) = E \left[e^{-\int_t^T R(X_u) du} g(X(T)) \mid X_t = x \right] \quad (4)$$

This theorem establishes a distinguished link between the analytical theory of PDEs and the probability theory relevant to SDEs. The most famous application of the above theorem is for the deriving of Black-Scholes (1973) formula when we let X_t be the stock price, $\mu(x, t) = rX_t$ and $g(X_T) = (S_T - K)^+$. In Black and Scholes' (1973) seminal paper, they solve the PDE via transforming it into a heat equation through a series of variable substitutions. A closed-form solution is derived in this way since the solution to a heat equation is well known in mathematical analysis. Yet we can now derive this closed-form formula in an easier and more intuitive way by applying the Feynman-Kac Formula. We are now ready to set up the pricing formulas for both default-free bonds and defaultable corporate bonds.

2.1 The Bond Market

Define $v(X_t, t)$ with $v \in C^{2,1}(\mathbb{R}^n \times [0, \infty))$ to be the price at the time of t of a zero-coupon bond maturing at time T . If no default occurs prior to the maturity date T , i.e. $T < \tau$, then $v(X_T, T) = 1$. The arbitrage-free price of a zero-coupon corporate bond conditioning on $t < \tau$ is:

$$v(X_t, t) = E_t \left[\exp \left(- \int_t^T R(X_u) du \right) G(\delta, T) \right] \quad (5)$$

In equation (5), $(1 - \delta)$ is the recovery rate at default and is conventionally assumed to be a constant between 0 and 1, whereas the function $G(\delta, T)$ necessitates some exposition. The existing literature generally assumes one of the following forms for $G(\delta, T)$.

- $G(\delta, T) = 1_{\{T < \tau\}} + (1 - \delta) 1_{\{\tau \leq T\}}$. This setting is called the **recovery of Treasury** scheme. At default, the creditors receive $(1 - \delta)$ fraction of an otherwise equivalent default-free bond, which eventually results in $(1 - \delta)$ at the maturity date. Examples assuming this scenario include Jarrow and Turnbull (1995), Cathcart and El-Jahel (1998), and Schobel (1999).
- $G(\delta, T) = 1_{\{T < \tau\}} + \frac{1 - \delta}{p(X_\tau, \tau)} 1_{\{\tau \leq T\}}$. This setting is called the **recovery of face value** scheme. At default the creditors receive $(1 - \delta)$ fraction of the face value, which is immediately reinvested in default-free bonds and eventually results in $\frac{1 - \delta}{p(X_\tau, \tau)}$ at the maturity date. This setting is adopted by Duffee (1998).
- $G(\delta, T) = \exp \left(- \int_t^T S(X_y) dy \right)$. There are two approaches resulting in this setting. The first approach is to take $S(x)$ as the short credit spread and directly model it as a non-negative stochastic process, e.g. Schmid and Zagst (2000). The second approach, adopted by Duffie and Singleton (1999), is called the **recovery of market value** scheme, and is dependent upon the condition $S(x) = \lambda(x)(1 - \delta)$, where $\lambda(x)$ is the intensity (hazard rate) function of default.

For other related information concerning recovery schemes, see Bielecki and Rutkowski (2002), or Duffie and Singleton (1999). In particular, the degenerate setting of $G(\delta, T) = 1$ corresponds to the risk-free bond and we rewrite equation (5) as

$$p(X_t, t) = E_t \left[e^{-\int_t^T R(X_u) du} \right] \quad (6)$$

Accordingly, the default-free short rate shown in equation (3) can now be expressed as:

$$R(x) = \lim_{T \rightarrow t^+} \frac{-1}{T-t} \ln p(x, t).$$

The following two sections demonstrate how the Feynman-Kac Formula (4) can considerably simplify the evaluation of defaultable bonds. In the first model, a closed-form formula is derived to replace Cathcart and El-Jahel's (1998) original numerical inversion of Laplace transformation for pricing defaultable bonds. In the second model, a simple expectation operation is used to replace Schobel's (1999) original procedure of employing the heat equation and the Green function.

3 Revisiting the Model of Cathcart and El-Jahel (1998)

Cathcart and El-Jahel (1998) propose a so-called "middle ground model" that lies between structural and reduced-form frameworks. It is a two-factor model that use the short rate $r(t)$ and a signaling variable $x(t)$ as the underlying state variables. Under the spot martingale measure Q , their dynamics are specified as.

$$dr(t) = \kappa(\mu - r(t))dt + \sigma_r \sqrt{r(t)} dW_1(t) \quad (7)$$

$$dx(t) = x(t)\alpha dt + x(t)\sigma_x \left(\rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t) \right) \quad (8)$$

where κ , μ , α , σ_r and σ_x are all positive constants, and ρ is a constant with absolute value less than 1. As claimed by Cathcart and El-Jahel (1998), the signaling process can capture a sample of effects that influences the probability of default. Also, the use of the signaling process is appropriate for issuers that do not have an identifiable collection of assets (for example, sovereign issuers and other agencies such as municipalities). First of all, the risk-free bond price $p(r, t)$ defined by equation (6) has a closed-form formula of CIR (1985) style after employing SDE (7) and formula (6). A default event is triggered once $x(t)$ hits the constant lower boundary x_1 .

Within the framework of this article, we have the following settings:

$$X_t = [\gamma(t)x(t)]^T \quad \text{and} \quad R(X_t) = \gamma(t)$$

Furthermore, the recovery of Treasury scheme is assumed:

$$G(\delta, T) = 1_{\{T < \tau\}} + (1 - \delta)1_{\{\tau \leq T\}}$$

Cathcart and El-Jahel (1998) solve the defaultable bond price $H(x, r, t)$ by specifying a two-variable PDE:

$$H_t + \kappa(\mu - r)H_r + \alpha x H_x + \frac{\sigma_r^2}{2} r H_{rr} + \rho \sqrt{r} x \sigma_r \sigma_x H_{rx} + \frac{\sigma_x^2}{2} x^2 H_{xx} = r H \quad (9)$$

The first boundary condition for PDE (9) is

$$v(x, r, T) = 1; \quad (10)$$

that is, at maturity date the bondholder gets the face value if no default has occurred. If x approaches infinity, then there is no chance of default and the bond's value approaches its corresponding default-free value. Hence, the second boundary condition is

$$v(\infty, r, t) = p(r, t). \quad (11)$$

When default occurs at time τ , i.e. $\tau = \inf \{t : x(t) = x_t\}$, the bondholder receive $1 - \delta$ of the value of the corresponding default-free bond according to the recovery of Treasury scheme. As a result, the third boundary condition is

$$v(x_t, r, \tau) = (1 - \delta)p(r, \tau) \quad (12)$$

To solve the above PDE, Cathcart and El-Jahel (1998) assume $\rho = 0$ and "guess" a solution of the form (equations (13) and (A-7) of their paper):

$$v(x, r, t) = p(r, t)(1 - \delta f(x, t)). \quad (13)$$

After substituting formula (13) into PDE (9), another simpler PDE for $f(x, t)$ is derived:

$$\frac{\sigma_x^2}{2} x^2 f_{xx} + \alpha x f_x + f_t = 0 \quad (14)$$

The required boundary conditions implied by (10), (11) and (12) are then, respectively,

$$f(x, t) = 0, \quad \lim_{x \rightarrow \infty} f(x, t) = 0, \quad \text{and} \quad f(x_t, \tau) = 1 \quad (15)$$

To solve equation (14), Cathcart and El-Jahel (1998) employ Laplace transformation technique and eventually express the solution in a so-called Bromwich integral with the

$$\text{form} \quad f(x, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{qt}}{q} \left(\frac{x}{x_t} \right)^{\lambda_1(q)} dq$$

Sophisticated numerical techniques are then called for. (Please refer to Appendix B of Cathcart and El-Jahel (1998) for other details.) Yet by exploiting the Feynman-Kac Formula, the following context will show that the above Bromwich integral can be replaced by the closed-form formula

$$f(x, t) = N \left(\frac{\ln \frac{x_l}{x} - \left(\alpha - \frac{\sigma_x^2}{2} \right) (T-t)}{\sigma_x \sqrt{T-t}} \right) + \left(\frac{x_l}{x} \right)^{\frac{2\alpha - \sigma_x^2}{\sigma_x^2}} N \left(\frac{\ln \frac{x_l}{x} + \left(\alpha - \frac{\sigma_x^2}{2} \right) (T-t)}{\sigma_x \sqrt{T-t}} \right) \quad (16)$$

where $N(\cdot)$ is the cumulative distribution function of a standard normal distribution. It is easy to see that closed-form formula (16) satisfies both PDE (14) and boundary conditions (15). Following PDE theory, expression (16) is indeed the unique solution to equation (14).

Why Cathcart and El-Jahel (1998) have been troubled by so much extra calculation is because they do not take advantage of the Feynman-Kac Formula. Formula (16) emerges from exploiting the Feynman-Kac Formula (4) such that the solution to PDE (9) can be derived as:

$$\begin{aligned} v(r, x, \tau) &= E_t \left[e^{-\int_t^\tau r(u) du} \left(\mathbf{1}_{\{\tau < T\}} + (1-\delta) \mathbf{1}_{\{\tau \leq T\}} \right) \right] \\ &= E_t \left[e^{-\int_t^\tau r(u) du} \right] E_t \left[\mathbf{1} - \delta \mathbf{1}_{\{\tau \leq T\}} \right] \\ &= p(r, t) \left(1 - \delta Q \left(\min_{t \leq s \leq T} x(s) \leq x_l \right) \right) \end{aligned} \quad (17)$$

In the above expressions, the second equality is due to Cathcart and El-Jahel's (1998) assumption that $\rho = 0$ in SDE (8). Therefore, the function of $r(t)$ and that of $x(t)$ are independent. The notation $Q \left(\min_{t \leq s \leq T} x(s) \leq x_l \right) = Q_t(\tau \leq T)$ represents the probability of default under the spot martingale measure; equivalently, it represents the probability that the signaling process $x(s)$ hits the default barrier x_l during the time interval t and T .

It is worth mentioning the following observation. When applying the Feynman-Kac Formula to Cathcart and El-Jahel's (1998) PDE (14), its solution is:

$$f(x, t) = E \left[\mathbf{1}_{\{\tau < T\}} \mid x(t) = x \right]$$

The maturity boundary condition $1_{\{t \leq T\}}$ is intuitively clear since the corporate defaults or not is completely verified at the maturity date. The computation of $Q\left(\min_{t \leq s \leq T} x(s) \leq x_t\right)$ is a standard first-passage-time problem of a geometrical Brownian motions and it allows for a closed-form formula. The computation is demonstrated in the Appendix.

In summary, the Feynman-Kac formula enables us to identify the pricing form (17) of the defaultable bond at the very beginning, instead of "guessing" its formula form. In addition, we are also immune from solving a PDE; therefore, we need not compute a numerical inversion of a Laplace transformation. Standard probability methodology grants us an analytical closed-form solution.

4 Revisiting the Model of Schobel (1999)

We briefly summarize the model of Schobel (1999) as follows. It is a two-factor model that use the short rate $r(t)$ and the corporate value $V(t)$ as the underlying state variables. Under the spot martingale measure Q , their dynamics are specified as:

$$dV(t) = V(t)r(t)dt + V(t)\sigma dW_1(t) \quad (18)$$

$$dr(t) = \kappa(\mu - r(t))dt + \eta\left(\rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t)\right) \quad (19)$$

The parameters σ , κ , μ and η are all positive constants, and ρ is a constant with absolute value less than 1. In other words, the corporate value $V(t)$ follows a geometrical Brownian motion and the short rate $r(t)$ follows a Ornstein-Uhlenbeck process, whereas ρ denotes the correlation coefficient between the two processes. Within the framework of this article, we have the following settings:

$$X_t = [r(t)V(t)]^T \quad \text{and} \quad R(X_t) = r(t)$$

Given the above assumptions, any claim $H(V, r, t)$ on the firm's assets with maturity T without interim cash-flows fulfills the following PDE:

$$\kappa(\mu - r)H_r + rV H_V + \frac{\sigma^2}{2}V^2 H_{VV} + \rho\sigma\eta H_{Vr} + \frac{\eta^2}{2}H_{rr} = rH \quad (20)$$

The corresponding boundary conditions are a maturity condition:

$$H(V, r, T) = 1$$

if $V(t) > V^*(t)$ for all $t \leq T$, and a pre-maturity boundary condition:

$$H(V^*, r, \tau) = (1 - \delta)p(r, \tau)$$

if $V(t) \leq V^*(t)$ for some $\tau \in [t, T]$, the risk-free bond price $p(r, t)$ also has a closed-form formula of Vasicek (1977) style after employing SDE (7) and formula (6). Define $V^*(t) \equiv Kp(r, t)$, then the default time of the corporate bond is

$$\tau = \inf\{s \in [t, T] : V(s) \leq Kp(r, s)\}. \quad (21)$$

Specifically, Schobel (1999) assumes that in case of default the corporate bond can be sold for $(1 - \delta)p(r(t), t)$ at any time $t \in [\tau, T]$. In other words, Schobel (1999) also adopts the recovery of Treasury scheme.

In order to solve the above PDE system, Schobel (1999) transform PDE (20) into a heat equation through a series of variable substitutions. Next, Schobel (1999) integrates the corresponding Green function and switches back to the original variables. Eventually, the solution is derived as:

$$v(V, r, t) = p(r, t)(1 - \delta f(V, r, t))$$

where

$$f(V, r, t) = N\left(\frac{\ln \frac{Kp(r, t)}{V} + \frac{1}{2}\Sigma^2}{\Sigma}\right) + \ln \frac{V}{Kp(r, t)} \left(\frac{\ln \frac{Kp(r, t)}{V} - \frac{1}{2}\Sigma^2}{\Sigma}\right) \quad (22)$$

and the transformed variable Σ is

$$\Sigma^2(t) \equiv \left(\sigma^2 + \frac{2\rho\sigma\eta}{\kappa} + \frac{\eta^2}{\kappa^2}\right)(T-t) - \left(\frac{2\rho\sigma\eta}{\kappa} + \frac{\eta^2}{\kappa^2}\right)B(t) - \frac{\eta^2}{2\kappa}B(t)^2 \quad (23)$$

And the function $B(t)$ is defined as:

$$B(t) \equiv \frac{1 - e^{-\kappa(T-t)}}{\kappa} \quad (24)$$

Now we demonstrate how to apply the Feynman-Kac formula to attain the above result. We apply the Feynman-Kac formula and the property of the recovery of Treasury scheme. The PDE (20) is then bounded only by the maturity condition:

$$v(V, r, T) = 1_{\{\tau > T\}} + (1 - \delta)1_{\{\tau \leq T\}}$$

Instead of using heat equation and Green function, we simply make use of the Feynman-Kac Formula and have the solution to PDE (20) as follows:

$$\begin{aligned}
v(V, r, t) &= E_t \left[e^{-\int_t^T r(u) du} \left(\mathbf{1}_{\{\tau > T\}} + (1 - \delta) \mathbf{1}_{\{\tau \leq T\}} \right) \right] \\
&= p(r, t) E_t \left[\frac{p(r(T), T)}{p(r, t)} e^{-\int_t^T r(u) du} (1 - \delta) \mathbf{1}_{\{\tau \leq T\}} \right] \\
&= p(r, t) E_t^{Q^T} [1 - \delta \mathbf{1}_{\{\tau < T\}}] \\
&= p(r, t) (1 - \delta f(V, r, t)) \tag{25}
\end{aligned}$$

where $f(V, r, t)$ has been defined in equation (22) and $p(r(T), T) = 1$ is the maturity value of the risk-free bond. Note that the third equality of the above equation is due to transforming the spot measure to the forward measure, and the last equation results from the first-passage-time problem of a geometric Brownian motion. We explicate them in details in the Appendix.

5 Conclusion

The Feynman-Kac Formula, named after Richard Feynman and Mark Kac, establishes a distinguishing link between partial differential equations and stochastic processes. It offers an intuitive approach to solve PDEs of the value of financial assets. Traditionally, it is used to model financial assets without default risk. This paper demonstrates that the Feynman-Kac Formula can also be used to derive the pricing formula of risky corporate bonds when the recovery scheme at default is assumed to be the recovery of Treasury scheme. In contrast, some researchers, e.g. Cathcart and El-Jahel (1998), evaluate the corporate bond by the numerical method of inverse Laplace transformation, whereas Schobel (1999) derive the closed-form pricing formula of corporate bonds by making use of the heat equation and the Green function. In the first model, a closed-form formula is derived to replace Cathcart and El-Jahel's (1998) original numerical inversion of Laplace transformation for pricing defaultable bonds. In the second model, a simple expectation operation is used to replace Schobel's (1999) original procedure of employing the heat equation and the Green function.

Appendix

The Fundamental Theorem of Asset Pricing

The fundamental theorem of asset pricing links the existence of an equivalent martingale measure to the no-arbitrage condition. It is originated by Cox and Ross' (1976) method of risk neutral valuation, which was formalized by Harrison and Kreps (1979).

Hence, under the framework of this paper, there exists a spot martingale measure Q such that for any asset process $V(t)$,

$$V(t)e^{-\int_0^t r(u)du} \text{ is a martingale almost surely.}$$

Its corresponding expectation operator is denoted as $E_t[\cdot]$ conditioning on the information up to time t . The asset value $\exp\left(\int_0^t r(u)du\right)$ is called the money account or the saving account, which acts as a numeraire for the spot martingale measure.

The Forward Martingale Measure

Similarly, we can define Q^T as the equivalent forward martingale measure such that

$$\frac{V_t}{p(r,t)} \text{ is a martingale } Q^T \text{ - almost surely.} \quad (26)$$

Its corresponding expectation operator is denoted as $E_t^{Q^T}[\cdot]$ conditioning on the information up to time t . The risk-free bond maturing at time T is used as a numeraire for the forward martingale measure. The Radon Nikodym derivative (Girsanov density) for transforming Q to Q^T is defined by

$$\frac{dQ^T}{dQ} = \frac{1}{P(r,t)} e^{-\int_t^T r(u)du}, \quad Q \text{ - almost surely.}$$

Note that the maturity value of the risk-free bond is $P(r(T), T) = 1$.

The First-Passage-Time Problem

Suppose the process $y(t)$ follows an arithmetic Brownian motion:

$$dy(t) = a dt + \sigma dW(t),$$

where $W(t)$ is a Wiener process under the martingale Q and $y(0) = 0$. if $-\infty < b < a < \infty$, we then have the following lemma:

$$Q(y(t) \leq a) = N\left(\frac{a - \alpha t}{\sigma \sqrt{t}}\right) \quad (27)$$

$$Q\left(y(t) > a, \min_{0 \leq s \leq t} x(s) \leq b\right) = e^{\frac{2b\alpha}{\sigma^2}} N\left(\frac{-a + 2b + \alpha t}{\sigma \sqrt{t}}\right) \quad (28)$$

Equation (27) is obvious, whereas equation (28) is derived through transforming the probability measure Q into another probability measure under which $y(t)$ is non-drifted. This is a well-known procedure, see, for example, Musiela and Rutkowski (1997) or Shreve (2004).

The Pricing Formula of Cathcart and El-Jahel (1998)

When $\rho = 0$, the dynamics of the logarithm of the signaling process (8) follows an arithmetic Brownian motion:

$$\ln x(t) = \left(\alpha - \frac{\sigma_x^2}{2}\right)dt + \sigma_x dW_2(t).$$

Based on the above formulas (27) and (28) and the initial condition $x(t) = x$, we can derive the default probability of equation (17) as follows:

$$\begin{aligned} Q\left(\min_{t \leq s \leq T} x(s) \leq x_l\right) &= Q(x(T) \leq x_l) + Q\left(x(T) > x_l, \min_{t \leq s \leq T} x(s) \leq x_l\right) \quad (29) \\ &= N\left(\frac{\ln \frac{x_l}{x} - \left(\alpha - \frac{\sigma_x^2}{2}\right)(T-t)}{\sigma_x \sqrt{T-t}}\right) + \left(\frac{x_l}{x}\right)^{\frac{2\alpha - \sigma_x^2}{\sigma_x^2}} N\left(\frac{\ln \frac{x_l}{x} + \left(\alpha - \frac{\sigma_x^2}{2}\right)(T-t)}{\sigma_x \sqrt{T-t}}\right) \end{aligned}$$

The above result is exactly the same as equation (16).

The Pricing Formula of Schobel (1999)

According to the definition of the default time τ in equation (21), we first have:

$$E_t^{Q^T} [1_{\{\tau \leq T\}}] = Q^T\left(\min_{t \leq s \leq T} \frac{V(t)}{p(r,t)} \leq T\right).$$

According to property (26) of the forward martingale measure, $\frac{V(t)}{p(r,t)}$ is a martingale

under Q^T . In particular, it is a Gaussian process with the dynamics:

$$d\left(\frac{V(t)}{p(r,t)}\right) = \frac{V(t)}{p(r,t)} \left((\sigma - \rho\eta B(t))dW_1(t) - \sqrt{1-\rho^2}\eta B(t)dW_2(t) \right),$$

where $B(t)$ is defined in equation (24). As a result, we see that the definition of Σ^2 in equation (23) is:

$$\Sigma^2 = \text{Var}\left(\int_t^T \ln \frac{V(u)}{p(r,u)} du\right)$$

Now, we can see that the default probability formula is quite similar to formula (29). As a matter of fact, we just replace variables of equation (29) as the following:

$$\alpha = 0, \quad \frac{x_t}{x} = \frac{K}{V}, \quad \sigma_x^2 = \Sigma^2$$

then we shall get formula $f(V, r, t)$ of equation (25).

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